INSTABILITIES OF THE DEGENERATE OPTICAL PARAMETRIC OSCILLATOR

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We study the stability of the degenerate optical parametric oscillator. The steady state is destabilized by a Hopf bifurcation. This Hopf is analyzed and the stability of the emerging periodic solutions is also determined analytically. Finally, we obtain numerically the upper boundary of stability of the periodic solutions. We find a destabilization of the periodic solutions either via a period-doubling bifurcation or via a direct transition to chaos.

1. Introduction

In this Communication we consider the problem of subharmonic conversion inside a resonant cavity pumped by an external laser. This device is also known as an optical parametric oscillator (OPO) when the subharmonic field has a finite intensity. The elementary process which takes place in the OPO is the absorption of one photon of the pump field at frequency 2ω and the emission of two photons at frequencies ω_1 and ω_2 with the energy conservation law $2\omega=\omega_1+\omega_2$. When $\omega_1=\omega_2$, the system is referred to as a degenerate optical parametric oscillator (DOPO), though the two beams can still be separated since they have different field polarizations. Since the early days of nonlinear optics, the OPO has been recognized as a model system for the generation of non-classical states of light [1-5]. Recently, the ENS group predicted [6,7] and demonstrated [8,9] that light squeezing can occur between the two emitted beams. Apart from its obvious interest in squeezing problems, the OPO is also of interest as a simple model for a nonequilibrium phase transition. This point of view motivated the research of McNeil et al. [10,11]. The DOPO model was recently investigated again [12] and it was shown that two disjoint domains of parameters exist: in one domain, the DOPO displays optical bistability, whereas in the other domain, a Hopf bifurcation to periodic solutions has been found. The purpose of this Communication is to study in more details this last domain of parameters for a DOPO.

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The semiclassical description of the DOPO is given by [10]

$$A_1' = -(1+i\Delta_1)A_1 + A_1^* A_0, \tag{1a}$$

$$A_0' = -(\gamma+i\Delta_0)A_0 - A_1^2 + E, \tag{1b}$$

where $A'=dA/dt$. We use normalized variables for the complex amplitude of the fundamental mode of the field A_0 inside the cavity, the subharmonic mode of the field A_1 and the input field amplitude E which is chosen as real and positive. The time is scaled to the cavity decay rate γ_1 of the subharmonic mode. The two detuning parameters are defined through

$$\Delta_1=(\omega_1-\omega)/\gamma_1, \quad \Delta_0=(\omega_0-2\omega)/\gamma_1,$$

where ω_0 and ω_1 are the cavity frequencies closest to 2ω and ω, respectively, with 2ω being the external field frequency. The reduced decay rate of the fundamental mode is $\gamma=\gamma_0/\gamma_1$.

The steady state solutions of eqs. (1) are

(i)

$$A_1=0, \quad |A_0|^2=E^2/(\gamma^2+\Delta_0^2), \tag{2}$$

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(ii)

$$|A_0|^2=1+\Delta_1^2,$$

$$E^2=|A_1|^4+2|A_1|^2(\gamma-\Delta_0\Delta_1)+(\gamma^2+\Delta_0^2)(1+\Delta_1^2). \quad (3)$$

2. Stability of the steady state solutions

As shown in ref. [12], the steady state solution (3) can display bistability when $\Delta_0\Delta_1>\gamma$. We shall restrict our analysis to the other domain defined by

$$\Delta_0\Delta_1<\gamma, \quad (4)$$

in which bistability between the steady solutions is impossible, but which was shown to be a necessary condition for the occurrence of a Hopf bifurcation point. Lugiato et al. [12] have determined that when the condition (4) holds there can be two bifurcation points:

1. When $E^2<E_c^2=(\gamma^2+\Delta_0^2)(1+\Delta_1^2)$, the solution (2) is stable and the solution (3) is unstable.

2. When $E^2<E_c^2$ the solution (2) is unstable and the solution (3) is stable, at least in the vicinity of the steady bifurcation point E_c.

3. When

$$\Delta_0\Delta_1<-(\gamma^2+2\gamma+\Delta_0^2)/2, \quad (5a)$$

$$|A_{1,H}|^2=-\frac{\gamma(\gamma^2+\Delta_0^2)[\Delta_0^2+(2+\gamma)^2]}{2(1+\gamma)^2(\Delta_0^2+2\Delta_0\Delta_1+\gamma^2+2\gamma)}, \quad (5b)$$

$$E_H^2=|A_{1,H}|^4+2|A_{1,H}|^2(\gamma-\Delta_0\Delta_1)+(\gamma^2+\Delta_0^2)(1+\Delta_1^2), \quad (5c)$$

there is a Hopf bifurcation at $E=E_H>E_c$ and the steady solution (3) looses its stability. At this point, a periodic solution emerges with a frequency Ω given by

$$\Omega^2=2|A_{1,H}|^2+(\gamma^2+\Delta_0^2)/(1+\gamma). \quad (6)$$

On fig. 1, we plotted E_H versus Δ_0 for different values of Δ_1 and with γ fixed to 1. The domain above the curves corresponds to the unstable steady solution.

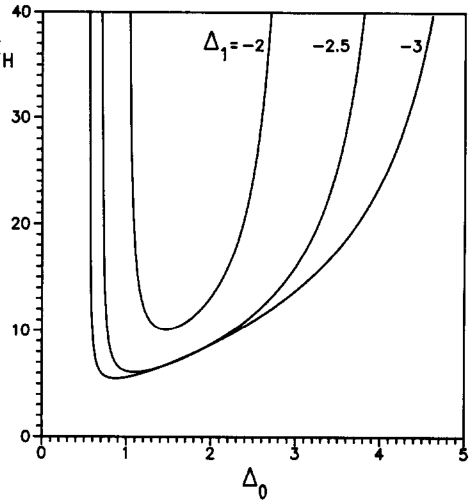

Fig. 1. Limits of stability of the steady state solutions. The boundaries correspond to the locus of Hopf bifurcations as given by eqs. (5) for $\gamma=1$.

3. Stability of the periodic solution

To determine the stability properties of the periodic solution which emerges at $E=E_H$, we make the usual local analysis in the vicinity of the Hopf bifurcation point. We first decompose the two complex field amplitudes into real and imaginary parts: $A_1=X+iU$ and $A_0=Y+iV$. They verify the equation

$$X'=-X+\Delta_1U+XY+UV,$$
$$U'=-U-\Delta_1X+XV-UY,$$
$$Y'=-\gamma Y+\Delta_0V-X^2-U^2+E,$$
$$V'=-\gamma V-\Delta_0Y-2XU. \quad (7)$$

In terms of these new variables, the steady state solutions are

$$X^2=(1+C/E)B/2,\quad U^2=(1-C/E)B/2,$$
$$Y=C/E\pm\Delta_1(1-C^2/E^2)^{1/2},$$
$$V=(C/E)\Delta_1\mp(1-C^2/E^2)^{1/2},$$

with $B=C-\gamma+\Delta_1\Delta_0$, $C^2=E^2-(\gamma\Delta_1+\Delta_0)^2$, and $C>0$.

Next, we introduce a smallness parameter ϵ through

$$|A_1|^2=|A_{1,H}|^2+\alpha\epsilon^2,\quad 0<\epsilon\ll1,\quad \alpha=O(1). \quad (8)$$

From this definition, we can determine $E^2=E_H^2+E_2\epsilon^2+O(\epsilon^4)$. In this simple case, it is easy to verify that the periodic solutions will be stable if E_2 is positive and unstable if E_2 is negative. To solve this problem, we proceed as follows. Let Z be any of the four dynamical variables X, Y, U or V. We seek solutions of eqs. (7) in the form

$$Z(t,\epsilon)=Z(T,\tau,\epsilon)$$
$$=\sum_{n=0}^{\infty}\epsilon^n Z(T,\tau,n)\equiv\sum_{n=0}^{\infty}\epsilon^n Z(n), \quad (9)$$

where $T=\omega(\epsilon)t=(\Omega+\epsilon^2\omega_2+...)t$ and $\tau=\epsilon^2 t$. In the decomposition (9), the zeroth order in ϵ is the steady solution at the Hopf bifurcation point which is obtained by combining (3) and (5). Near the Hopf bifurcation, an infinitesimal perturbation of the steady state displays, at least for a finite time, an oscillatory evolution towards its final state. Below (above) the Hopf bifurcation point, the time T characterizes the time-scale of the oscillations frequency, whereas the time τ characterizes the time-scale of the oscillations damping (divergence). Inserting (9) into (7), we seek 2π-periodic solutions on the time scale T: $Z(T,\tau,\epsilon)=Z(T+2\pi,\tau,\epsilon)$ and treat the two times T and τ as independent variables:

$$Z'\equiv \mathrm{d}Z/\mathrm{d}t=\partial Z/\partial T+\epsilon^2\partial Z/\partial\tau\equiv Z_T+\epsilon^2 Z_\tau .$$

To first order in ϵ, we obtain from eq. (7) a homogeneous system of linear differential equations which can be written as

$$\boldsymbol{B}(1)_T=L\boldsymbol{B}(1)$$

for the column 4-vector $\boldsymbol{B}(n)\equiv\boldsymbol{B}(T,\tau,n)=\mathrm{col}[X(n),Y(n),U(n),V(n)]$. All higher order equations are linear differential inhomogeneous equations:

$$\boldsymbol{B}(n)_T=L\boldsymbol{B}(n)+\boldsymbol{C}(n),$$

where the column 4-vector $\boldsymbol{C}(n)$ depends on the components of the vectors $\boldsymbol{B}(n')$ with $n'<n$. We define the scalar product for 4-vectors through

$$\langle\boldsymbol{u}|\boldsymbol{v}\rangle=\sum_{n=1}^{4}u_n v_n^* .$$

With this definition, the adjoint M of the matrix L is given by $\langle M\boldsymbol{u}|\boldsymbol{v}\rangle=\langle\boldsymbol{u}|L\boldsymbol{v}\rangle$. Let $\boldsymbol{w}$ be the eigenvector of the adjoint matrix M associated with the eigenvalue $\mathrm{i}\Omega$. Then the Hopf theorem applied to this problem proves that the solution $\boldsymbol{B}(1)$ is bounded and 2π-periodic in T if the following solvability condition is verified [13]:

$$\langle\boldsymbol{w}|\boldsymbol{C}(3)\rangle=0. \quad (10)$$

This condition is a complex equation for the two unknowns $E_2=F(\gamma,\Delta_0,\Delta_1)$ and $\omega_2=G(\gamma,\Delta_0,\Delta_1)$. Of special interest is therefore the condition $F(\gamma,\Delta_0,\Delta_1)=0$ which determines the boundary for supercritical Hopf bifurcations, i.e. for stable small amplitude periodic solutions near the bifurcation point. Although an explicit analytic expression can be given for this boundary, it is far too complicated to be of any use. We have therefore represented graphically the boundary $F=0$ in the (Δ_0,Δ_1) plane for $\gamma=1$ with the curve drawn in fig. 2 with a full line. However, the knowledge of the boundary $E_2=0$ is not sufficient. Indeed, in order for the series expansion of E in powers of ϵ to be uniform, a necessary condition is that $E_2=\mathrm{O}(1)$. To check this property, we have added on fig. 2 lines corresponding to the solution of $E_2=F(\gamma,\Delta_0,\Delta_1)=\pm0.2$ and ±0.4. When these contour lines nearly coincide, E_2 varies rapidly and quickly reaches the domain where it is an O(1) function. In this domain, the perturbation expansion developed in this section may be convergent and the boundary $E_2=0$ bears a useful information. On the

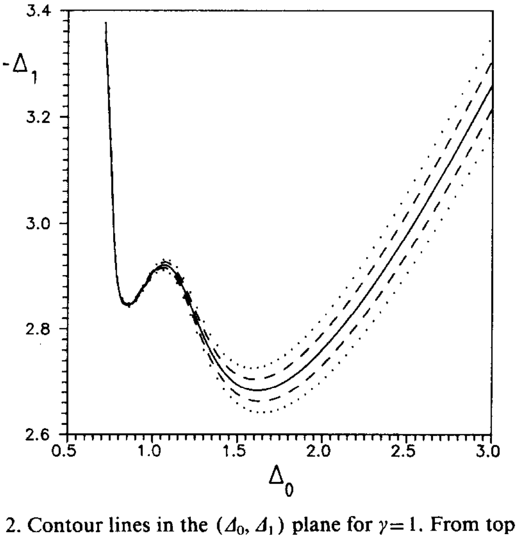

Fig. 2. Contour lines in the (Δ_0,Δ_1) plane for $\gamma=1$. From top to bottom, the five lines are the solutions of $E_2=F(\gamma,\Delta_0,\Delta_1)=0.4$, 0.2, 0.0, -0.2 and -0.4, respectively. For each set of parameters, the domain above the contour line is a domain of supercritical Hopf bifurcation.

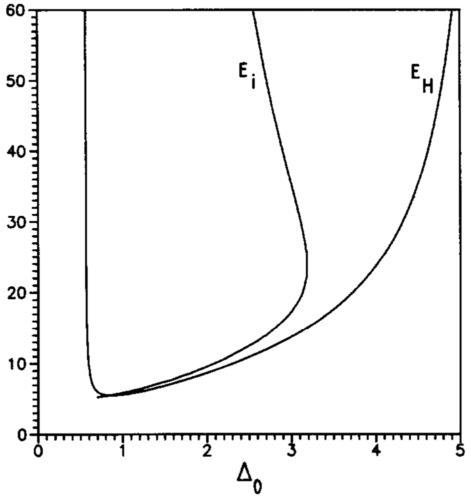

Fig. 3. Boundaries for the stability of the steady state solution (E_H) and the periodic solution (E_i) for $\gamma=1$ and $\Delta_1=-3$.

contrary, when the contour lines are well separated, E_2 increases only very slowly from zero to an O(1) value. When this is the case, there is a large domain around $E_2=0$ where the expansion (9) is not uniform and the analysis of this section is invalidated. Another way to understand this problem is to consider the variation of the subharmonic field intensity as a function of the input field intensity variation:

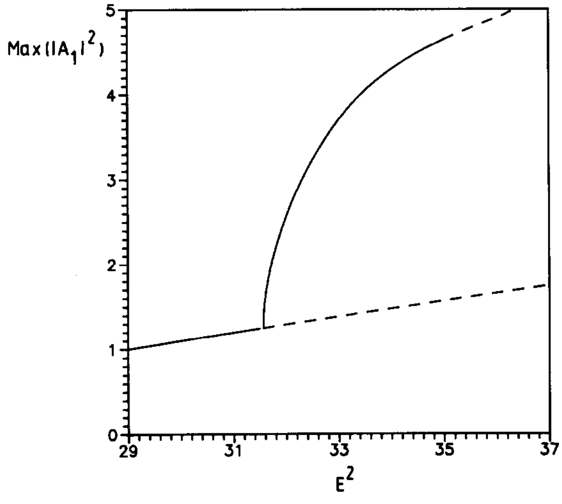

Fig. 4. Example of bifurcation diagram with a supercritical Hopf bifurcation at $E_H=5.62$ for $\gamma=1$, $\Delta_0=1$ and $\Delta_1=-3$. The graph represents the maximum of the subharmonic oscillation amplitude versus the input field intensity. Full lines correspond to stable solutions. The periodic solution becomes unstable at $E=5.907$ via a period doubling bifurcation.

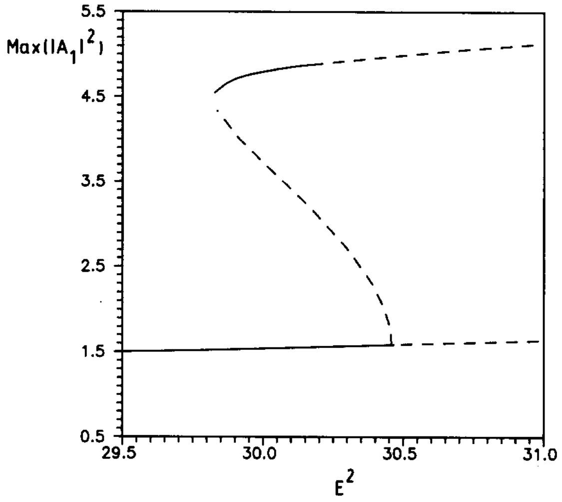

Fig. 5. Example of bifurcation diagram with a subcritical Hopf bifurcation at $E_H=5.519$ for $\gamma=1$, $\Delta_0=0.83$ and $\Delta_1=-3$. The hysteresis exists in the domain $5.46<E<5.519$. The stable periodic solution on the upper branch becomes unstable at $E=5.493$.

$$|A_1|^2=|A_{1,H}|^2+\alpha(E^2-E_H^2)/E_2. \tag{11}$$

Hence, we see that, when E_2 is of O(1), a variation of E^2 induces a comparable variation of $|A_1|^2$. On the contrary, when $E_2 \ll O(1)$ a small variation of E^2 induces a large variation of $|A_1|^2$.

4. Numerical solutions

In order to go beyond the analysis of the vicinity of the Hopf bifurcation point, we have analyzed eqs. (7) with the AUTO programme [14]. All numerical results reported in this section have been obtained for $\gamma=1$ and $\Delta_1=-3$. Fig. 3 gives a summary of the results obtained in this way. The curve labelled E_H is the locus of the Hopf bifurcation points as given by eqs. (5). The second curve corresponds to the limit of stability of the periodic solution which emerges from the Hopf bifurcation point. As seen on fig. 3, the two curves cross for $\Delta_{0,c}\simeq 0.85$. When $\Delta_0<\Delta_{0,c}$, the Hopf bifurcation is subcritical. Otherwise, it is supercritical and the area between the two curves on fig. 3 gives the domain of stability of this periodic solution. On fig. 4, we display an example of a supercritical Hopf bifurcation. The periodic solution becomes unstable through a period doubling bifurcation. On the contrary, on fig. 5, we display an example of a subcritical Hopf bifurcation. As expected this case leads to bistability. The intermediate branch is of course unstable but only a small fraction of the upper branch shows stable oscillations. Beyond the domain of stability of these periodic solutions, chaos has been found. The parameters used to construct the bifurcation diagram of fig. 5 give the additional information that in the (Δ_0, Δ_1) plane of fig. 2, the domain of supercriticality which lies above the curve $E_2=0$ can be fairly narrow. Hence, the DOPO may be rather sensitive to fluctuations when operating beyond the steady state regime.

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